Off-Shell Duality in Born-Infeld Theory

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Abstract

The classical equations of motion of Maxwell and Born-Infeld theories are known to be invariant under a duality symmetry acting on the field strengths. We implement the $SL(2,\mathbb{Z})$ duality in these theories as linear but non-local transformations on the potentials. We show that the action and the partition function in the Hamiltonian formalism are modular invariant in any gauge. For the Born-Infeld theory we find that the longitudinal part of the fields have to be complexified.

I. INTRODUCTION

Duality plays a fundamental role in describing the same physical system using different variables. It provides a valuable tool to understand different aspects of the same theory. For instance, the five perturbative string theories are now known to be related to each other through a series of different dualities. Also, D-branes are known to be solitonic objects in string theory and as such they carry information about its non-perturbative sector. When string or M-theory is consistently truncated the resulting quantum field theory also presents some duality which is reminiscent from that of the larger theory. Some supersymmetric gauge theories inherit such dualities as, for instance, the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. A similar situation occurs for D-branes whose low energy limit is described by a Born-Infeld theory. In particular, the D3-brane action of type IIB string theory is $SL(2, \mathbb{Z})$ self-dual and this symmetry is inherited by the four dimensional Born-Infeld theory [1]. Usually the original theory and the dual one are distinct, but sometimes the dual theory coincides with the original one. We will concentrate on such theories in this paper.

In field theory duality is often realized as symmetries of the equations of motion. It is well known that the equations of motion, Bianchi identities and the energy-momentum tensor of Maxwell and Born-Infeld theories are invariant under $SO(2)$ rotations which mix the electric and magnetic fields [2, 3]. The action, however, is not invariant. The $SO(2)$ symmetry can be enlarged to $SL(2, \mathbb{R})$ when a dilaton and an axion are added [3, 4]. This motivated the search for gauge theories whose equations of motion are duality invariant.

As stated before, duality is found as symmetries of the classical equations of motion and energy-momentum tensor. However, it is desirable that such symmetries could be implemented at the quantum level as symmetries of the action and partition function. In this way the symmetries will hold in any situation and not only for on-shell quantities. This is relevant, for instance, in the derivation of Ward identities among the off-shell Green's functions. Off-shell symmetries must be implemented in the basic field variables (and not in the field strengths for gauge theories) either in the Lagrangian or Hamiltonian formalism.

When varying the action the resulting boundary term must be local in time, giving rise to a Noether current associated to the invariance. However, the boundary term can be non-local in space provided that it has a sufficient falloff at spatial infinity. This allows the variations of the basic field variables to be non-local in space. These ideas were first explored in [5] where the $SO(2)$ symmetry of Maxwell equations were implemented at the action level in the Hamiltonian formalism in Coulomb gauge. The transformations of the vector potential and its canonical momentum are non-local in space

$$\delta A_i = \alpha \epsilon_{ijk} \frac{\partial^j}{\nabla^2} E^k, \qquad \delta E_i = \alpha \epsilon_{ijk} \partial^j A^k. \tag{1.1}$$

On-shell they give rise to the usual $SO(2)$ transformation between the electric and magnetic fields $\delta E_i = \alpha B_i, \delta B_i = -\alpha E_i$. The corresponding Noether charge, which generates the rotation, is also non-local and has an expression involving Chern-Simons terms [5]. The same holds for the Born-Infeld theory [6] and for gauge theories coupled to matter and gravity [7].

The transformations Eqs.(1.1) leave the action invariant only in the Coulomb gauge. This could be seen as a drawback since the symmetry manifests itself only in a particular gauge. Even so, it may be quite useful. A typical example is the Chern-Simons theory in Landau gauge. In this case there appears a vector supersymmetry [8] which can be extended to the exceptional algebra $D(2, \alpha)$ [9]. This symmetry is essential to show the renormalizability of the model.

We should point out that there is an alternative procedure to implement off-shell symmetries in the action with local transformation laws. Usually it breaks manifest Lorentz invariance and demands the introduction of more fields. For the Maxwell theory this requires a description in terms of two potentials giving rise to the Schwarz-Sen model [10] or, alternatively, an infinite number of them [11]. Duality manifests itself as rotations between the potentials. It is possible to show that the duality symmetry of the Schwarz-Sen model is the local form of the non-local transformations Eqs.(1.1) [12]. Although the Schwarz-Sen model is not manifestly Lorentz covariant this symmetry can be made manifest by the in-

clusion of auxiliary fields and some gauge symmetry through the PST formalism [13]. A similar situation is found for the Born-Infeld theory [14]. It should be remarked that this situation is not exclusive of duality symmetry. Even well known symmetries, like the BRST symmetry, can be cast into a non-local form at the expense of loosing manifest Lorentz invariance [15]. In this work, however, we shall not follow this approach.

The $SL(2,\mathbb{R})$ symmetry of the equations of motion found when a dilaton and an axion are added, manifests, at the quantum level, as an $SL(2,\mathbb{Z})$ duality of the partition function. This happens when the dilaton and the axion take their vacuum expectation value which are combined into a complex coupling constant τ with its real part being the theta term. Now the action and the partition function are functions of τ and duality manifests as modular transformations of the coupling constant[1]. There are two basic ways to implement duality in gauge theories. In the first one a new gauge field is introduced in such a way that upon functional integration over the original field a dual theory is obtained [16]. The second way treats duality as a canonical transformation of the original phase space variables [17]. In this paper we will concentrate in this second approach since it can be connected with the classical symmetries of the equations of motion.

In Maxwell theory the Lagrangian partition function is found to be a modular form under $SL(2,\mathbb{Z})$ transformations of the coupling constant τ [16, 18]. The weights of the modular transformation are proportional to the Euler number and the signature of the space-time. At the Hamiltonian level, the partition function is modular invariant with modular weight equal to zero [17, 18]. In this case duality can be implemented as a canonical transformation on the reduced phase space. This means that Gauss law holds and we are on-shell. Also, the canonical transformation has essentially the form Eq.(1.1) and holds only in Coulomb gauge.

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[1]A function $F(\tau)$ is said to be a modular form of weight (u,v) if, under a modular transformation $\tau \to \tilde{\tau}$, Eq.(3.2) below, it transforms as $F(\tilde{\tau}) = (c\tau + d)^u (c\tilde{\tau} + d)^v F(\tau)$.

In this paper we will show how the non-local $SO(2)$ transformations Eqs.(1.1) can be extended to $SL(2, \mathbb{Z})$ transformations for the Maxwell and Born-Infeld theories with a theta term. We will also show that duality holds off-shell in the sense that Gauss law is not required. It holds also in any gauge, and not just in Coulomb gauge as originally proposed in [5]. We will find that the generalization of Eqs.(1.1) to the $SL(2, \mathbb{Z})$ case is also non-local in space. To study duality at the quantum level we consider the phase space partition function and the Hamiltonian BRST formalism. Since the Born-Infeld theory is non-renormalizable we treat it as an effective field theory and its partition function should be considered in this context. We will show that the $SL(2, \mathbb{Z})$ transformations can be regarded as a canonical transformation and that the phase space partition function is modular invariant. We also find that in order to implement a linear $SL(2, \mathbb{Z})$ transformation for the Born-Infeld theory it is necessary to consider the longitudinal part of the fields as being complex.

II. $SO(2)$ DUALITY IN MAXWELL THEORY

Maxwell theory with a theta term is described by the following action in Minkowski space with metric $(+ - --)$

$$S = -\frac{1}{8\pi} \int d^4x \ \left(\frac{4\pi}{g^2} F^{\mu\nu} F_{\mu\nu} + \frac{\theta}{2\pi} F^{\mu\nu *} F_{\mu\nu} \right), \tag{2.1}$$

where $^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. The Hamiltonian formulation is obtained in a straightforward way. There is a primary constraint $\Pi^0 = \frac{\delta L}{\delta \dot{A}_0} = 0$ and the secondary constraint receives no contribution from the theta term, giving rise to the usual Gauss law $\partial_i \Pi^i = 0$. The Hamiltonian density is then

$$H_M = -\frac{2\pi i}{\tau - \overline{\tau}} \Pi^i \Pi_i - i \frac{\tau + \overline{\tau}}{\tau - \overline{\tau}} \Pi^i B_i - \frac{i}{2\pi} \frac{\tau \overline{\tau}}{\tau - \overline{\tau}} B^i B_i, \tag{2.2}$$

where $\Pi^i = \frac{\delta L}{\delta \dot{A}_i}$, the magnetic field is $B_i = \frac{1}{2}\epsilon_{ijk} F^{jk}$ and the complex coupling constant is $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$. The contribution from the theta term appears in the second and third terms of Eq.(2.2).

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The BRST charge is the same as in pure Maxwell theory since the constraint structure was not modified

$$Q = \int d^3x \ \left(\partial_i \Pi^i C + \mathcal{P}_D \Pi_0 \right). \tag{2.3}$$

The ghosts obey the canonical Poisson brackets $\{\mathcal{P}_C, C\} = \{\mathcal{P}_D, D\} = -1$ and the BRST transformations are

$$\delta A_i = \partial_i C, \qquad \delta A_0 = -\mathcal{P}_D, \qquad \delta \Pi_0 = 0, \qquad \delta \Pi_i = 0,$$

$$\delta C = 0, \qquad \delta \mathcal{P}_C = -\partial_i \Pi^i, \qquad \delta D = -\Pi_0, \quad \delta \mathcal{P}_D = 0. \tag{2.4}$$

The partition function is then

$$Z(\tau) = \int \mathcal{D}A_\mu \ \mathcal{D}\Pi_\nu \ \mathcal{D}(\text{ghosts}) \ e^{-iS_M(\tau)}, \tag{2.5}$$

where the Maxwell effective action is

$$S_M(\tau) = \int d^4x \ \left(\Pi^\mu \dot{A}_\mu + \dot{C}\mathcal{P}_C + \dot{\mathcal{P}}_D D - H_M - \{Q, \Psi\} \right), \tag{2.6}$$

and Ψ is the gauge fixing function.

As discussed before the Maxwell theory without a theta term and in Coulomb gauge has an $SO(2)$ duality symmetry acting on the potentials [5, 12]. To go to the Coulomb gauge we choose $\Psi = \frac{1}{\epsilon}\partial_i A^i D + A_0 \mathcal{P}_C$, perform the field transformation $\Pi_0 \to \epsilon\Pi_0$, $D \to \epsilon D$, which has the Jacobian equal to one, and take the limit $\epsilon \to 0$. After integration over $\mathcal{P}_C, \mathcal{P}_D, A_0$ and Π_0 the partition function Eq.(2.5) reduces to

$$Z(\tau) = \int \mathcal{D}A_i \ \mathcal{D}\Pi_i \ \mathcal{D}C \ \mathcal{D}D \ \delta(\partial_i A^i) \ \delta(\partial_i \Pi^i) \ \exp[-i \int d^4x \ (\Pi^i \dot{A}_i - H_M + D\partial^2 C)], \quad (2.7)$$

where $\partial^2 = \partial^i \partial_i$. For the pure Maxwell theory without a theta term the infinitesimal $SO(2)$ duality transformations which leave the partition function invariant are[2]

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[2]These transformations differ from those of Eqs.(1.1) by 2π factors. That is due to different normalizations for the action.

$$\delta\Pi_i = \frac{\alpha}{2\pi}B_i, \qquad \delta A_i = -2\pi\alpha\epsilon_{ijk}\frac{\partial^j}{\partial^2}\Pi^k,$$

$$\delta C = \delta D = 0. \tag{2.8}$$

On shell they reduce to the usual $SO(2)$ rotation between the electric and magnetic fields. They also commute with the BRST transformations. They are local in time but non-local in space. An important assumption which was implicitly taken is that the coupling constant g is invariant under duality. In fact, in [5, 12] g^2 was taken to be equal to 2.

It is easy to verify that with a theta term the partition function looses its invariance under duality. In fact, the Hamiltonian is no longer invariant. Besides that there are other annoying points. The first one is that when $\theta = 0$ duality holds only in Coulomb gauge. A second point is that the coupling constant changes under duality [16] and if we wish to implement the symmetry on the potentials it should act also on the coupling constants. However this was not taken into account in [5, 12]. The third point regards the symmetry group. There is, in fact, an $SL(2,\mathbb{Z})$ symmetry and not just an $SO(2)$ symmetry when a theta term is present [16]. This raises the question whether it would be possible to implement an $SL(2,\mathbb{Z})$ symmetry on the potentials as it was done for $SO(2)$ rotations.

III. $SL(2,\mathbb{Z})$ DUALITY IN MAXWELL THEORY

In order to consider the $SL(2,\mathbb{Z})$ duality it proves to be convenient to split the vector fields A_i and Π_i into their transversal A_i^T, Π_i^T and longitudinal parts A_i^L, Π_i^L. We will also consider finite $SL(2,\mathbb{Z})$ transformations. We have found that the $SL(2,\mathbb{Z})$ transformations are given by

$$A_i^T = a\tilde{A}_i^T + 2\pi c\epsilon_{ijk}\frac{\partial^j}{\partial^2}\tilde{\Pi}^{Tk}, \qquad \Pi_i^T = d\tilde{\Pi}_i^T + \frac{b}{2\pi}\tilde{B}_i,$$

$$A_i^L = |a - c\tilde{\tau}|\tilde{A}_i^L, \qquad \Pi_i^L = \frac{1}{|a - c\tilde{\tau}|}\tilde{\Pi}_i^L,$$

$$A_0 = |a - c\tilde{\tau}|\tilde{A}^0, \qquad \Pi_0 = \frac{1}{|a - c\tilde{\tau}|}\tilde{\Pi}_0,$$

$$C = |a - c\tilde{\tau}|\tilde{C}, \qquad \mathcal{P}_C = \frac{1}{|a - c\tilde{\tau}|}\tilde{\mathcal{P}}_C,$$

$$D = \frac{1}{|a - c\tilde{\tau}|}\tilde{D}, \qquad\qquad \mathcal{P}_D = |a - c\tilde{\tau}|\tilde{\mathcal{P}}_D, \qquad\qquad (3.1)$$

$$\tilde{\tau} = \frac{a\tau + b}{c\tau + d}, \qquad\qquad\qquad (3.2)$$

where a, b, c and d are integers satisfying $ad - bc = 1$.[3] Notice that the transversal (or physical) part of the vectors are transformed among themselves while the gauge dependent (or non-physical) parts, which include: A_0, A_i^L, Π_0 and Π_i^L and the ghosts, transform into themselves. These transformations are local in time but non-local in space.

The Hamiltonian density Eq.(2.2) is not invariant under Eqs.(3.1,3.2). It transforms as

$$H_M = \tilde{H}_M - \frac{2\pi i}{\tilde{\tau} - \overline{\tilde{\tau}}} \frac{2(a - |a - c\tilde{\tau}|) - c(\tilde{\tau} + \overline{\tilde{\tau}})}{|a - c\tilde{\tau}|} \tilde{\Pi}^{Ti}\tilde{\Pi}_i^L$$
$$- \frac{i}{\tilde{\tau} - \overline{\tilde{\tau}}} \frac{(a - |a - c\tilde{\tau}|)(\tilde{\tau} + \overline{\tilde{\tau}}) - 2c\tilde{\tau}\overline{\tilde{\tau}}}{|a - c\tilde{\tau}|} \tilde{\Pi}^{Li}\tilde{B}_i, \qquad (3.3)$$

and upon integration the extra terms give rise to surface contributions. The kinetic terms in the effective action Eq.(2.6) are also invariant up to surface terms. Hence, the Hamiltonian is modular invariant up to surface terms.

The gauge fixing term in Eq.(2.6) requires some care. It is easy to show that the BRST charge Eq.(2.3) is invariant under duality. Hence, it follows that Ψ must be modular invariant in order that the gauge fixing term in Eq.(2.6) remains modular invariant. Consider first the most general expression for Ψ which implements linear gauge choices. We write Ψ as

$$\Psi = \int d^3x \left(\chi D + A_0 \mathcal{P}_C \right), \qquad\qquad (3.4)$$

with χ depending only on the gauge dependent pieces. If we require that Ψ be modular invariant under duality then χ must transform as $\chi = |a - c\tilde{\tau}|\tilde{\chi}$. The most general expression for χ linear in the gauge dependent pieces and with the correct transformation property is

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[3]If we consider just the classical theory the condition that a, b, c and d are integers can be relaxed and the duality group is then $SL(2, \mathbb{R})$.

$$\chi = \alpha \partial_i A^{Li} + \frac{\beta}{\tau - \bar{\tau}} \Pi_0 + \gamma A_0 + \frac{\delta}{\tau - \bar{\tau}} \partial^i \Pi_i^L, \tag{3.5}$$

with α, β, γ and δ arbitrary numbers. Therefore any linear gauge is modular invariant. This includes, among others, the Coulomb gauge for which $\chi = \frac{1}{\epsilon} \partial^i A_i^L$ and covariant gauges for which $\chi = \frac{\xi}{2} \frac{\Pi_0}{\tau - \bar{\tau}} + \partial^i A_i^L$. It is easy to generalize the above argument for nonlinear gauge choices. We have just to multiply the fields in χ by appropriate powers of $\tau - \bar{\tau}$ so that the product transforms with a power of $|a - c\tilde{\tau}|$. Hence any gauge choice can be made modular invariant.

We then conclude that the effective action is modular invariant for any gauge choice. The Jacobian of the transformations Eqs.(3.1) can be computed and it is found to be equal to one. Therefore, they can be regarded as a canonical transformation. Hence the path integral measure is also invariant. As a consequence, the partition function is modular invariant. It should be stressed that the partition function in the Lagrangian formalism is not modular invariant under duality, rather it transforms as a modular form [16]. However, the phase space partition function is modular invariant [17, 18].

Finally we must show that Eqs.(3.1) reduce to the familiar duality transformations of the classical equations of motion. They are given by [4]

$$G_{\mu\nu} = a\tilde{G}_{\mu\nu} + \frac{b}{2\pi} {}^*\tilde{F}_{\mu\nu},$$
$$F_{\mu\nu} = c {}^*\tilde{G}_{\mu\nu} - \frac{d}{2\pi}\tilde{F}_{\mu\nu}, \tag{3.6}$$

where $G_{\mu\nu} = -2\frac{\partial L}{\partial F^{\mu\nu}}$. Also, g^2 and θ are identified with the vacuum expectation values of the dilaton ϕ and axion a, respectively, as

$$\frac{1}{g^2} = \frac{<e^{-\phi}>}{2}, \qquad \frac{\theta}{4\pi^2} = - <a> . \tag{3.7}$$

For the Maxwell theory with a theta term we have that

$$G_{\mu\nu} = -\frac{1}{g^2}F_{\mu\nu} - \frac{\theta}{8\pi^2} {}^*F_{\mu\nu}, \tag{3.8}$$

and we can check that Eqs.(3.6) are indeed valid when Eqs.(3.1) are used on-shell, that is, when Gauss law holds. Also, the complex combination of the axion and dilaton $a - ie^{-\phi}$

transforms as τ, as it should. Therefore, the duality transformations Eqs.(3.1) are the off-shell version of Eqs.(3.6).

At the classical level we can reduce Eqs.(3.1) to $SO(2)$ transformations by choosing $a = d = \cos\alpha$ and $b = -c = \sin\alpha$. For infinitesimal transformations we find that

$$\delta A_i^T = -2\pi\alpha\epsilon_{ijk}\frac{\partial^j}{\partial^2}\Pi^k, \qquad \delta A_i^L = \frac{1}{2}\alpha(\tau + \overline{\tau})A_i^L,$$

$$\delta\Pi_i^T = \frac{\alpha}{2\pi}B_i, \qquad \delta\Pi_i^L = \frac{1}{2}\alpha(\tau + \overline{\tau})\Pi_i^L,$$

$$\delta A_0 = \frac{1}{2}\alpha(\tau + \overline{\tau})A_0, \qquad \delta\Pi_0 = \frac{1}{2}\alpha(\tau + \overline{\tau})\Pi_0, \tag{3.9}$$

while the real and imaginary parts of τ, respectively τ_R and τ_I, transform as

$$\delta\tau_R = -\alpha(1 + \tau_R^2 - \tau_I^2), \qquad \delta\tau_I = -2\alpha\tau_R\tau_I. \tag{3.10}$$

The transformations for the gauge dependent pieces are proportional to θ while the transversal parts have the usual non-local transformations. When $\theta = 0$ the gauge dependent pieces are invariant and Eq.(3.10) fixes the imaginary part of τ as $\frac{4\pi}{g^2} = 1$. Then the transversal parts have the usual transformations Eqs.(2.8).

IV. BORN-INFELD THEORY

It is well known that the Born-Infeld theory has an $SO(2)$ symmetry in its classical equations of motion which can be extended to $SL(2, \mathbb{R})$ if an axion and a dilaton are added [4]. If we consider just the axion and dilaton vacuum expectation values we get a Born-Infeld theory with a theta term. With the identifications made in Eqs.(3.7) its action is

$$S = \int d^4x \left(1 - \frac{\theta}{16\pi^2}F^{\mu\nu*}F_{\mu\nu} - \sqrt{1 + \frac{1}{g^2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4g^4}(F^{\mu\nu*}F_{\mu\nu})^2}\right). \tag{4.1}$$

In the weak field limit it reduces to Maxwell theory with a theta term Eq.(2.1). There is also a dimensionful constant in the action which was set equal to one. In order to handle the square root in the action we introduce an auxiliary field V

$$S = \int d^4x \left[1 - \frac{\theta}{16\pi^2}F^{\mu\nu*}F_{\mu\nu} - \frac{V}{2}\left(1 + \frac{1}{g^2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4g^4}(F^{\mu\nu*}F_{\mu\nu})^2\right) - \frac{1}{2V}\right]. \tag{4.2}$$

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The Hamiltonian formulation is straightforward and follows closely that of [14]. Since we have introduced an auxiliary field V there are two primary constraints $\Pi_0 = 0$ and $p = \frac{\partial L}{\partial V} = 0$. From the first constraint we get as secondary constraint the Gauss law. From the second constraint we get an algebraic equation for V which can be solved so that V is eliminated. We then find the Hamiltonian density

$$H_{BI} = \sqrt{1 + 2H_M - (\Pi^i B_i)^2 + B^i B_i \Pi^j \Pi_j} - 1. \tag{4.3}$$

Clearly the Hamiltonian is not modular invariant under Eqs.(3.1,3.2). The Maxwell Hamiltonian is not invariant and since the duality transformations are linear the extra terms in Eq.(3.3) can not be canceled against those coming from $(\Pi B)^2 - B^2 \Pi^2$ term in the square root in Eq.(4.3). Either non-linear terms must be introduced in Eqs.(3.1) or something else must be modified.

It must be noted that both the Maxwell and the Born-Infeld Hamiltonian densities can be rewritten in terms of a complex vector field

$$P_i = \Pi_i + \frac{\tau}{2\pi} B_i. \tag{4.4}$$

We find that

$$H_M = -\frac{2\pi i}{\tau - \overline{\tau}} P^i \overline{P}_i, \tag{4.5}$$

and

$$H_{BI} = \sqrt{1 - \frac{4\pi i}{\tau - \overline{\tau}} P^i \overline{P}_i - \frac{4\pi^2}{(\tau - \overline{\tau})^2} (P \times \overline{P})^2} - 1, \tag{4.6}$$

where the overline denotes complex conjugation. The vector P_i transforms under duality as

$$P_i = \frac{1}{a - c\tilde{\tau}} \left(\tilde{\Pi}_i^T + \frac{a - c\tilde{\tau}}{|a - c\tilde{\tau}|} \tilde{\Pi}_i^L + \frac{\tilde{\tau}}{2\pi} \tilde{B}_i \right), \tag{4.7}$$

while

$$\frac{1}{\tau - \overline{\tau}} = \frac{|a - c\tilde{\tau}|^2}{\tilde{\tau} - \overline{\tilde{\tau}}}. \tag{4.8}$$

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This explains why the Maxwell Hamiltonian is not invariant. The longitudinal and transversal parts of $\tilde{\Pi}_i$ do not combine themselves back into $\tilde{\Pi}_i$ so that P_i is not a modular form. If instead of $|a - c\tilde{\tau}|$ in the denominator of the $\tilde{\Pi}_i^L$ term in Eq.(4.7) we had just $a - c\tilde{\tau}$ we could recover $\tilde{P}_i$. But taking out the modulus in the transformations Eqs.(3.1) is not consistent because all fields are real. On the other side if we could change only the transformation for Π_i^L that would do the job. It is then necessary that Π_i^L possess an imaginary part. For consistency A_0, Π_0, A_i^L and the ghosts must have an imaginary part as well.

So we start with the non-physical sector $A_0, \Pi_0, A_i^L, \Pi_i^L$ and the ghosts all described by complex fields. Since the number of ghosts has also doubled the number of physical degrees of freedom is still the same. The vectors A_i and Π_i are now complex with their transversal part taken to be real while their longitudinal parts are taken to be complex. The effective action is now

$$S_{BI} = \int d^4x \ \left(\frac{1}{2}\overline{\Pi}^\mu \dot{A}_\mu + \frac{1}{2}\Pi^\mu \dot{\overline{A}}_\mu + \frac{1}{2}\dot{C}\overline{\mathcal{P}}_C + \frac{1}{2}\dot{\overline{C}}\mathcal{P}_C + \frac{1}{2}\dot{\mathcal{P}}_D\overline{D} + \frac{1}{2}\dot{\overline{\mathcal{P}}}_D D - H_{BI} - \{Q, \Psi\}\right),$$

$$(4.9)$$

The Hamiltonian density has the same form as in Eq.(4.6) with P_i defined by Eq.(4.4) but with complex fields instead of real fields. The integrand in the square root in Eq.(4.6) is real.

The BRST charge is now

$$Q = \frac{1}{2}\int d^3x \ \left(\partial_i \Pi^i \overline{C} + \partial_i \overline{\Pi}^i C + \overline{\mathcal{P}}_D \Pi_0 + \mathcal{P}_D \overline{\Pi}_0 \right),$$

$$(4.10)$$

so that Q is real. The BRST transformations are modified in a straightforward way. The gauge fixing fermion reads now

$$\Psi = \frac{1}{2}\int d^3x \ \left(\chi \overline{D} + \overline{\chi}D + A_0 \overline{\mathcal{P}}_C + \overline{A}_0 \mathcal{P}_C \right),$$

$$(4.11)$$

and is also real.

Now we have to show that this theory is equivalent to the original Born-Infeld theory. In order to do that we will perform a partial gauge fixing so that all imaginary parts are

gauged away. Let us denote the real and imaginary parts of any complex field φ as φ_R and φ_I, respectively. Let us choose the imaginary part of χ as $\chi_I = \frac{1}{\epsilon}\partial_i A_I^{Li}$ and assume that χ_R does not depend on Π_{0I} and A_{0I}. Let us perform the transformation $\Pi_{0I} \to \epsilon\Pi_{0I}$, $D_I \to \epsilon D_I$, whose Jacobian is equal to one. When the limit $\epsilon \to 0$ is taken the effective action Eq.(4.9) reduces to

$$
\begin{aligned}
S_{BI} = \int d^4x \; (\Pi_{0R}\dot{A}_{0R} + \Pi^{Ti}\dot{A}_i^T + \Pi_R^{Li}\dot{A}_{iR}^L + \Pi_I^{Li}\dot{A}_{iI}^L + \dot{C}_R\mathcal{P}_{CR} + \dot{C}_I\mathcal{P}_{CI} + \dot{\mathcal{P}}_{DR}D_R - \mathcal{P}_{DR}\mathcal{P}_{CR} \\
- \mathcal{P}_{DI}\mathcal{P}_{CI} - \delta\chi_R D_R - C_I\partial^2 D_I - \Pi_{0R}\chi_R - \Pi_{0I}\partial_i A_I^{Li} + A_{0R}\partial_i\Pi_R^{Li} + A_{0I}\partial_i\Pi_I^{Li} - H_{BI}), \quad (4.12)
\end{aligned}
$$

where $\delta\chi_R$ is the BRST transformation of χ_R. Now let us perform the integral over the imaginary part of all fields. The integration over Π_{0I} produces a delta functional $\delta(\partial_i A_I^{Li})$ and since the longitudinal part of A_I^i has just one component this means that $A_I^{Li} = 0$. Then the integration over A_I^{Li} can be performed as well. The same is true for the integration over A_{0I}. It gives $\Pi_I^{Li} = 0$ and the integration over Π_I^{Li} can also be performed. At this stage H_{BI} depends only on the transversal components A_i^T, Π_i^T and the real part of the longitudinal components A_{iR}^L, Π_{iR}^L and reduces to Eq.(4.3). We now integrate over the ghosts. The integration over $\mathcal{P}_{CI}$ produces $\delta(\dot{C}_I - \mathcal{P}_{DI})$ and the integration over $\mathcal{P}_{DI}$ sets $\mathcal{P}_{DI} = \dot{C}_I$. The remaining integrations over C_I and D_I gives a $\det \partial^2$ which can be absorbed into the path integral normalization. Then, only the real part of the ghosts remain in the effective action and they are the ghost contribution that we would get if we had started with all fields real. So we have shown that there is a gauge choice which eliminates completely the imaginary part of all fields and that the resulting theory is the original Born-Infeld theory.

Then the $SL(2,\mathbb{Z})$ duality transformations are now

$$
\begin{aligned}
A_i^T &= a\tilde{A}_i^T + 2\pi c\epsilon_{ijk}\frac{\partial^j}{\partial^2}\tilde{\Pi}^{Tk}, & \Pi_i^T &= d\tilde{\Pi}_i^T + \frac{b}{2\pi}\tilde{B}_i, \\
A_i^L &= (a - c\bar{\tilde{\tau}})\tilde{A}_i^L, & \Pi_i^L &= \frac{1}{a - c\tilde{\tau}}\tilde{\Pi}_i^L, \\
A_0 &= (a - c\bar{\tilde{\tau}})\tilde{A}^0, & \Pi_0 &= \frac{1}{a - c\tilde{\tau}}\tilde{\Pi}_0, \\
C &= (a - c\bar{\tilde{\tau}})\tilde{C}, & \mathcal{P}_C &= \frac{1}{a - c\tilde{\tau}}\tilde{\mathcal{P}}_C, \\
D &= \frac{1}{a - c\tilde{\tau}}\tilde{D}, & \mathcal{P}_D &= (a - c\bar{\tilde{\tau}})\tilde{\mathcal{P}}_D. & (4.13)
\end{aligned}
$$

The unphysical sector is composed of modular forms. The vector P_i is also a modular form. It transforms as

$$P_i = \frac{1}{a - c\tilde{\tau}}\tilde{P}_i, \tag{4.14}$$

so that the Maxwell Hamiltonian is modular invariant with no surface terms being generated. The Born-Infeld Hamiltonian is also modular invariant. It is easy to show that the kinetic terms in the effective action Eq.(4.9) are also invariant up to surface terms. The BRST charge Eq.(4.10) is also invariant. By an argument similar to that presented in Section III we conclude that the gauge fixing term in Eq.(4.9) is also modular invariant so that the effective action Eq.(4.9) is modular invariant. Finally we can show that the duality transformations Eqs.(4.13) have a unity Jacobian so that the partition function is modular invariant.

The duality transformations Eqs.(4.13) reduce to the usual duality transformations of the classical equations of motion Eqs.(3.6). Now the expression for $G_{\mu\nu}$ is much more complicated because it involves a square root. However it is straightforward to show that Eqs.(3.6) hold when use is made of Gauss law.

V. CONCLUSION AND DISCUSSION

We have shown how it is possible to generalize the $SL(2, \mathbb{R})$ symmetry of the equations of motion, for Maxwell and Born-Infeld theories, to an off-shell duality. For the Maxwell theory we found that the Hamiltonian H_M is modular invariant up to a surface term. In the Born-Infeld case it was necessary to consider the longitudinal part of the fields as complex fields. Then the Born-Infeld Hamiltonian H_{BI} is strictly modular invariant with no boundary terms being generated by the transformation. Of course, we could consider Maxwell theory with the longitudinal part of the fields being complex as well. In this case the Hamiltonian would be modular invariant without any boundary term. However there is no clear interpretation for the complex longitudinal fields introduced in these theories.

Another important question is whether we can extend the symmetry to the case where the axion and the dilaton are propagating fields since it is known that the equations of motion have an $SL(2,\mathbb{R})$ symmetry [4]. In this case τ is no longer constant but a field whose vacuum expectation value is given by Eq.(3.7). The action is then $S = S_0 + S_{BI}$ where

$$S_0 = -2 \int d^4x \frac{\partial^\mu \tau \partial_\mu \overline{\tau}}{|\tau - \overline{\tau}|^2}. \tag{5.1}$$

It is easy to show that S_0 is indeed duality invariant. The Hamiltonian is not modified since no integration by parts was done and it remains invariant under duality. However, the kinetic terms in Eq.(4.9) are no longer invariant because now τ is time (and space) dependent. In fact, only the non-physical sector looses the invariance while the physical one remains invariant.

Since self-dual theories, as those studied here, are endowed with special properties it would be interesting to find the supersymmetric extension of the off-shell duality transformations. There is an intimate connection between self-duality and spontaneous symmetry breaking of supersymmetry [19] and knowing the duality transformations may help to elucidate this relationship.

It would be also interesting to study the noncommutative case. Since $SL(2,\mathbb{Z})$ is a nonperturbative symmetry of type IIB string theory we expect that it should be relevant in the noncommutative case as well [20]. However, it seems that the noncommutative gauge theory obtained from the D3-brane with a B-field along the brane is no longer self-dual [21].

It is also known that the equations of motion of p-forms have a duality symmetry [17]. It would be interesting to find the extension of our transformations to that case. Another interesting question is whether our non-local transformations can be made local along the lines of [12].

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VI. ACKNOWLEDGMENTS

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